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Sound production at the edge of a steady flow

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The theory initiated by Lighthill (1952) to describe the sound radiated by turbulence embedded in an uniform fluid at rest is here extended to the case where the turbulence exists on the edge of a uniformly moving stream. An exact analogy is developed between the distant real sound field and that which would be radiated by a particular quadrupole distribution adjacent to a vortex sheet positioned in the linearly disturbed flow. The equivalent sources in this analogy are quadrupoles identical in strength with those in Lighthill's model, but the quadrupoles are now shown to convect with the fluid-particle velocity. There is no amplifying effect of shear. The particular case of a plane shear layer is worked out in detail for sound waves of scale large in comparison with the shear-layer thickness.

A downstream zone of silence is predicted as is the formation of highly directional beams associated with the interference of sound radiated directly and sound reflected from the fluid interface. A distinct structure results in which the variation of sound with flow velocity, density and angle is not easily accounted for by simple power-law scaling. Finally a comparison is made with some features of jet noise; the modelling of the high frequency jet noise problem by a single shear layer yields some features consistent with experiment.

1. A development of the acoustic analogy

This paper concerns the sound radiated by a source at or near the edge of an extensive steady stream embedded in otherwise still fluid. We seek to determine the influence of that stream on the radiated sound. The source may be a physical inhomogeneity, or turbulence produced at the unstable interface between the two regions. That problem models jet noise production in those regions where the turbulent mixing layer is thin on the jet scale, as it inevitably is close enough to the nozzle exit.

Sound propagates through the still fluid of density ρ_0 with sound speed c_0 according to the linear wave equation

$$\Box^2\rho = \partial^2\rho/\partial t^2 - c_0^2\nabla^2\rho = 0. \tag{1}$$

The exact equations of fluid motion are combined in Lighthill's (1952) inhomogeneous wave equation

$$\Box^2\rho = \partial^2 T_{ij}/\partial x_i \partial x_j, \tag{2}$$

$$T_{ij} = \rho u_i u_j + p_{ij} - c_0^2(\rho-\rho_0)\,\delta_{ij}. \tag{3}$$

p_{ij} is the difference between the compressive stress tensor and its mean value in the quiescent medium. With this definition T_{ij} is set to be zero in the wave field, so that (2) describes how the nonlinear and inhomogeneity-induced terms drive the acoustic field in precisely the same manner as a quadrupole distribution would drive an ideal acoustic medium at rest. Once T_{ij}, the quadrupole strength, is known, the sound field is known also and once the nonlinear terms are determined, so is the wave field that they drive. In those cases where sound is generated by low Mach number turbulence which exists essentially independently of the small compressibility effects, the stress tensor can be specified as if the fluid were incompressible. Lighthill's theory then provides a complete description of the sound field generated by that turbulence to which sound represents an essentially negligible by-product.

But it is only when the turbulence is slow in the sense that the r.m.s. fluctuation in Mach number is small and retarded time differences across the eddy correlation scale are negligible (the eddy is compact) that the sound can be described in this straightforward way. Even slow turbulence, if sufficiently extensive, can generate a strong field which must eventually influence the turbulence stress tensor (Crighton 1969). Neither, as Lighthill explained in his pioneering paper, can the stress tensor associated with small regions of sufficiently rapid turbulence be specified independently of the sound. When the turbulence is neither slow nor compact, the task of specifying the stress tensor acquires much of the inevitable complexity of fully nonlinear unsteady compressible flows. Crow (1970) voiced a commonly held doubt that the aerodynamic sound problem would then admit a 'cause and effect' ordering. At high Mach number the subject is probably inextricable from that of 'compressible turbulence'.

But a few situations in which the stress tensor is very extensively distributed are tractable. For example, the propagation of sound through turbulence can be treated by Lighthill's method and this has been done by Lighthill (1953), Crow (1969) and Ffowcs Williams & Howe (1973). The interaction terms are estimated from sound and turbulence which to first order exist independently. The scattering sources inevitably contain secular terms that drive the secondary waves at a condition of resonance. These must be treated carefully and recognized to reflect the fact that waves travel at a speed slightly different from c_0 when nonlinearities are admitted, or when propagating through inhomogeneous fluid. The source field has therefore to be described very accurately and this is only possible of course in those relatively simple flows that are properly understood.

Nonlinear propagation of a one-dimensional wave can be similarly treated as a superposition of a first-order field and secondary waves generated by the Lighthill stress tensor estimated from first-order theory (Ffowcs Williams 1973). Again the inevitable secular terms arise and have to be treated carefully, which they can be only when, as in that problem, the interaction process is very clearly and completely defined.

Lighthill (1952) emphasized that the linearized side of the equation has to be chosen to be the correct left-hand side in any 'extensive' region where the waves propagate differently from those in the ambient medium at rest. If this is not done, the difficulty of using the Lighthill method is made obvious by considering the two identical equations

$$\partial^2\rho/\partial t^2 - c_0^2\nabla^2\rho = Q \tag{4}$$

and

$$\partial^2\rho/\partial t^2 - c^2\nabla^2\rho = Q + (c_0^2 - c^2)\nabla^2\rho. \tag{5}$$

These equations can both be 'solved' to give

$$\rho(\mathbf{x},t) = \frac{1}{4\pi c_0^2}\int [Q]\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}$$
$$= \frac{1}{4\pi c^2}\int\left[Q + (c_0^2 - c^2)\frac{\partial^2\rho}{\partial y_i^2}\right]\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c}\right)\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{6}$$

From these identical but apparently radically different expressions it is clear that the linear term in the second form must contain important features that have to be expressed properly before the solution is evident. The second form is in fact an integral equation which has to be solved for ρ, and this is characteristically the way in which propagation effects are modelled in the Lighthill analogy.

In our problem we wish to consider the influence on the radiation field of two distinct regions with different steady states. In the Lighthill analogy these could again be modelled through extensively distributed linear terms in the stress tensor, but we hesitate to try that approach because as we have already said the effects are likely to be evident only when a precise description of the stress tensor is available. By its nature, turbulence and in all probability turbulence-induced sound will never be subject to such a revelation!

We expect major changes to arise from these effects and that significant cases are not necessarily confined to those where inhomogeneities occur on the direct propagation path. For example, a monopole source near the plane interface between semi-infinite volumes of light and heavy fluid with identical sound speeds radiates preferentially into the light fluid regardless of its location provided that it is closer to the interface than a fraction of a wavelength. When the density ratio is large, the influence of the light fluid is to transform the field in the heavy fluid from monopole to dipole, even when the monopole source is located in the heavy medium.

To bring out major effects such as these we take a more direct approach than that of attempting to interpret the influence of subtle linear 'source' terms. We choose instead to describe the field in terms of sources that are confined to a small region of space but which are supplemented by linear boundary terms. Those terms are distributed over a control surface situated beyond (when viewed from the still fluid) the source region according to Curle's (1955) extension of Lighthill's theory:

$$4\pi c_0^2|\mathbf{x}|H(\rho-\rho_0)(\mathbf{x},t) \sim \frac{\partial^2}{\partial t^2}\int_V \frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}[T_{ij}]\,d^3\mathbf{y}$$
$$-\frac{\partial}{\partial t}\int_S \frac{l_j}{c_0}\left[\frac{x_i}{|\mathbf{x}|}(p_{ij}+\rho u_i u_j) + \rho c_0 u_j\right]d^2\mathbf{y} \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{7}$$

The geometry is indicated in figure 1. H is the Heaviside function equal to unity when $\mathbf{x}$ lies in the volume V and zero otherwise. The surface S that bounds V

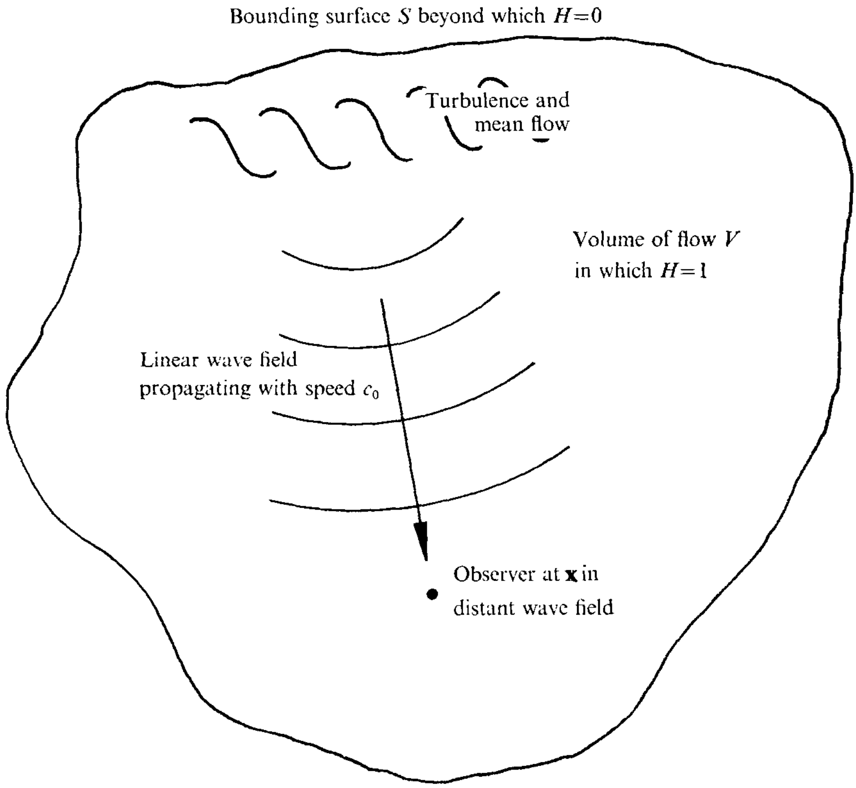


FIGURE 1. Diagram illustrating the relative positions of the turbulent flow, the control surface S, the volume V and the distant observation point $\mathbf{x}$.

may sometimes be made up of many parts, but we shall concentrate later on the special case where S comprises the plane $x_3 = 0$. $|\mathbf{x}-\mathbf{y}|$ is the distance separating the source point $\mathbf{y}$ from an observer at $\mathbf{x}$ and l_i are the direction cosines of the outward normal from V at S. Square brackets indicate that the function they enclose is to be evaluated at the source position $\mathbf{y}$ at the retarded time $t-|\mathbf{x}-\mathbf{y}|/c_0$.

Our object now is to eliminate all linear terms from the distributed volume source, and to cater especially for the case when the turbulence is formed on a thin shear layer on which most of the vorticity is concentrated. Then, as Lighthill showed, the stress tensor is likely to be dominated by a linear term

$$\partial(\rho u_i u_j)/\partial t,$$

which is equivalent to the product of the pressure and the mean rate-of-strain tensor. We prefer not to regard this type of linear term as known and feel that to do so might very well prove misleading for two reasons. First, as it is linear, there is no guarantee that it is free from secular terms capable of completely transforming the general character of the solution. Second, since concentrated vorticity tends to move with the fluid particles, large pointwise time derivatives can be induced by relatively steady convection of that vorticity, a process we think unlikely to constitute an effective source of sound, but perfectly capable of giving confusingly high values to the partial time derivatives of the stress tensor. Those derivatives must then actually constitute a higher order and less effective source field which integrates to zero in the limit of zero Mach number.

There is in this subject a clear precedent in which pressure terms initially thought to represent the dominant aerodynamic source actually integrate to nearly zero. The pressure at a solid plane surface supporting a turbulent boundary layer was for some time thought to constitute a powerful aerodynamic dipole. But Powell (1960) showed how, when the issue is examined with adequate care, the dipoles are seen to be arranged in a self-cancelling quadrupole array. In fact Powell showed how pressure terms at a plane solid surface merely account for specular reflexion of the aerodynamic quadrupoles. The arguments used by Ffowcs Williams (1965*b*) to show that the surface pressure generated by turbulence near a plane cannot constitute anything other than a quadrupole field, though developed for a rigid boundary, actually hold true in a more general context and apply to any plane homogeneous surface. Such surfaces are equally incapable of supporting strong sources. They act as mere reflectors (Ffowcs Williams 1965*a*).

Is it possible that the source field identified by Lighthill as being amplified by shear and likely to constitute in jets the dominant longitudinal quadrupole actually degenerates to an essentially weaker field? We shall show that this is indeed the case, and we anticipate the result here by noting that the formal limit under which the shear term is selected by Lighthill (1954) as dominant is when the mean velocity gradient tends to infinity. Since the shear layer bounds a stream of finite velocity, this limit corresponds to a vortex sheet in which the source density $\partial(\rho u_i u_j)/\partial t = p\,\partial\overline{U}(y)/\partial y$ (when i and j correspond respectively to the direction of the stream $\overline{U}$ and its gradient, y) actually equals the volume source term $p\,\overline{U}_\infty \delta(y-y_0)$, $\overline{U}_\infty$ being the velocity difference across the plane shear layer centred at y_0 and δ Dirac's delta function. This limit causes, therefore, the volume distribution to collapse into a surface term and Lighthill's shear-amplified source is actually $\overline{U}_\infty$ times Curle's rigid-surface source, the effective vanishing of which aroused so much interest in the past.

To bring out this effect in a specific way, we re-arrange the volume source term to express it in terms of Lagrangian time derivatives $D/Dt = \partial/\partial t + u_i \partial/\partial x_i$, which cannot receive a misleadingly powerful contribution from the *convection* of concentrated vorticity. We do this by means of the technique described by Ffowcs Williams (1966), and write

$$\rho \frac{D}{Dt}\left\{\frac{G}{\rho} T_{ij}\right\} = \frac{\partial}{\partial t}\{GT_{ij}\} + \frac{\partial}{\partial x_k}\{u_k G T_{ij}\}. \tag{8}$$

G in this expression could actually be *any* continuous function of space-time, but it is sometimes helpful to regard it as the generalized instantaneous Doppler factor based on the moving fluid particle,

$$G = (1-M_r)/\{(1-M_r)^2 + \epsilon^2 M^2\}, \tag{9}$$

where $M = |\mathbf{u}|/c_0$, $M_r = u_i x_i/c_0|\mathbf{x}|$ and ϵ is a similarly arbitrary function which we shall initially suppose to be the local turbulence intensity but later set to zero everywhere in a more formal identification of the source field as quadrupoles moving with the unsteady fluid velocity.

Equation (8) is now integrated over the volume V at the retarded time $t-|\mathbf{x}-\mathbf{y}|/c_0$. The second term on the right-hand side can then be split into two parts, viz.

$$\int_V \left[\frac{\partial}{\partial y_k}(u_k G T_{ij})\right] d^3\mathbf{y} \sim \int_V \frac{\partial}{\partial y_k}[u_k G T_{ij}]\, d^3\mathbf{y} - \frac{\partial}{\partial t}\int_V [M_r G T_{ij}]\, d^3\mathbf{y} \quad (|\mathbf{x}| \to \infty)$$
$$= \int_S l_k [u_k G T_{ij}]\, d^2\mathbf{y} - \frac{\partial}{\partial t}\int_V [M_r G T_{ij}]\, d^3\mathbf{y}. \tag{10}$$

Inclusion of this identity in the retarded time integral of (8) leads to the following relationship between elements of the distant radiation field:

$$\frac{\partial}{\partial t}\int_V [T_{ij}]\, d^3\mathbf{y} = \int_V \left[\rho \frac{D}{Dt}\left\{\frac{G}{\rho} T_{ij}\right\}\right] d^3\mathbf{y} - \int_S l_k [u_k G T_{ij}]\, d^2\mathbf{y}$$
$$+ \frac{\partial}{\partial t}\int_V [\{1 - G(1-M_r)\} T_{ij}]\, d^2\mathbf{y}, \tag{11}$$

or

$$\frac{\partial^2}{\partial t^2}\int_V [T_{ij}]\, d^3\mathbf{y} = \frac{\partial}{\partial t}\int_V \left[\rho \frac{D}{Dt}\left\{\frac{G}{\rho} T_{ij}\right\}\right] d^3\mathbf{y} - \frac{\partial}{\partial t}\int_S l_k [u_k G T_{ij}]\, d^2\mathbf{y}$$
$$+ \frac{\partial^2}{\partial t^2}\int_V \left[\frac{\epsilon^2 M^2 T_{ij}}{\{(1-M_r)^2 + \epsilon^2 M^2\}}\right] d^3\mathbf{y}. \tag{12}$$

This can be rewritten when (11) is regarded as a reduction formula as

$$\frac{\partial^2}{\partial t^2}\int_V [T_{ij}]\, d^3\mathbf{y} = \int_V \left[\rho \frac{D}{Dt}\left\{G \frac{D}{Dt}\left(\frac{G T_{ij}}{\rho}\right)\right\}\right] d^3\mathbf{y} - \int_S l_k \left[u_k G \rho \frac{D}{Dt}\left\{\frac{G}{\rho} T_{ij}\right\}\right] d^2\mathbf{y}$$
$$- \frac{\partial}{\partial t}\int_S l_k [u_k G T_{ij}]\, d^2\mathbf{y} + \frac{\partial^2}{\partial t^2}\int_V \left[\frac{\epsilon^2 M^2 T_{ij}}{(1-M_r)^2 + \epsilon^2 M^2}\right] d^3\mathbf{y}$$
$$+ \frac{\partial}{\partial t}\int_V \left[\frac{\epsilon^2 M^2}{(1-M_r)^2 + \epsilon^2 M^2} \rho \frac{D}{Dt}\left\{\frac{G}{\rho} T_{ij}\right\}\right] d^2\mathbf{y}. \tag{13}$$

This equation allows an unambiguous interpretation of the volume source terms since there is now no possibility of the apparent source-strength density reaching unreasonably high local transient values as strong elements of vorticity (or density gradient) are 'silently' convected past the point in question. Of course if such elements are convected rapidly enough, so that $M_r = 1$, for example, they are extremely noisy and the fourth term on the right-hand side of (13) then re-emphasizes them. Also, if they are accelerating rapidly enough or the flow is locally chaotic and noisy then $\epsilon^2 M^2$ or its time derivative can again be big enough to re-emphasize Lighthill's shear-amplified source as a truly dominant one. But often, the theory will be applied to flows of low Mach number and low turbulence level where $\epsilon^2 M^2 \ll 1$ and the only surviving shear-amplified term is then extremely weak, weaker in fact than the octupoles discarded by Lighthill in arriving at his result.

Equation (13) now allows (7) to be rewritten in an exact far-field form of Curle's equation:

$$4\pi c_0^2|\mathbf{x}|H(\rho-\rho_0)(\mathbf{x},t) = \frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}\int_V \left[\rho \frac{D}{Dt}\left\{G\frac{D}{Dt}\left(\frac{G}{\rho}T_{ij}\right)\right\}\right] d^3\mathbf{y}$$
$$-\frac{\partial}{\partial t}\int_S l_i[\rho_0 G u_i]\, d^2\mathbf{y} - \frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}\int_S l_k\left[u_k G\rho\frac{D}{Dt}\left\{\frac{G}{\rho}T_{ij}\right\}\right] d^2\mathbf{y}$$
$$+\frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}\frac{\partial^2}{\partial t^2}\int_V\left[\frac{\epsilon^2 M^2 T_{ij}}{(1-M_r)^2+\epsilon^2M^2}\right] d^3\mathbf{y}$$
$$-\frac{\partial}{\partial t}\int_S l_i[u_i\rho\{1-(1-M_r)G\}\{1+M_r\}]\, d^2\mathbf{y}$$
$$-\frac{x_i}{|\mathbf{x}|c_0}\frac{\partial}{\partial t}\int_S l_k\left[p_{ik}+Gp_{ij}u_k\frac{x_j}{|\mathbf{x}|c_0}\right] d^2\mathbf{y}$$
$$+\frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}\frac{\partial}{\partial t}\int_V\left[\frac{\epsilon^2M^2}{(1-M_r)^2+\epsilon^2M^2}\rho\frac{D}{Dt}\left\{\frac{G}{\rho}T_{ij}\right\}\right] d^3\mathbf{y}. \quad (14)$$

Now suppose that the volume V contains all the important source terms and that the surface S is situated in and lies parallel to the linearly disturbed uniform flow in which viscosity is negligible. Only the linear surface terms in (14) need then be retained to supplement the volume sources. Also since ϵ is zero on the surface G reduces there to $(1-M_r)^{-1}$, and

$$\frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}\frac{\partial^2}{\partial t^2}\int_V\left[\frac{\epsilon^2M^2T_{ij}}{(1-M_r)^2+\epsilon^2M^2}\right] d^3\mathbf{y} = \int_V\left[\frac{\partial^2}{\partial y_i\partial y_j}\left\{\frac{\epsilon^2M^2T_{ij}}{(1-M_r)^2+\epsilon^2M^2}\right\}\right] dV.$$

Equation (14) then simplifies considerably to give

$$4\pi c_0^2|\mathbf{x}|H(\rho-\rho_0)(\mathbf{x},t) = T_+ - \frac{\rho_0}{(1-M_r)}\frac{\partial}{\partial t}\int_S l_i[u_i]\, d^2\mathbf{y} - \frac{x_i}{|\mathbf{x}|}\frac{1}{c_0}\frac{\partial}{\partial t}\int_S l_i[p]\, d^2\mathbf{y}, \quad (15)$$

where,

$$T_+ = \frac{x_i x_j}{|\mathbf{x}|^2c_0^2}\int_V\left[\rho\frac{D}{Dt}\left\{G\frac{D}{Dt}\left(\frac{G}{\rho}T_{ij}\right)\right\}\right] d^3\mathbf{y} + \int_V\left[\frac{\partial^2}{\partial y_i\partial y_j}\{T_{ij}[1-(1-M_r)G]\}\right] d^3\mathbf{y}$$
$$-\frac{x_i}{|\mathbf{x}|}\frac{1}{c_0}\int_V\left[\frac{\partial}{\partial y_j}\left\{\frac{\epsilon^2M^2}{(1-M_r)^2+\epsilon^2M^2}\rho\frac{D}{Dt}\left(\frac{G}{\rho}T_{ij}\right)\right\}\right] d^3\mathbf{y}, \quad (16)$$
$$G = (1-M_r)/\{(1-M_r)^2+\epsilon^2M^2\}.$$

Equation (15) is an integral equation for the field, the surface integrands both being terms linearly related to the density perturbation.

At low Mach numbers only the first term in T_+ need be retained, the others being smaller by a factor of order ϵ^2M^2. But at high Mach numbers critical layers arise (where $M_r = 1$) and then only the second and third terms in (16) need be retained.

The parameter ϵ has been held finite so far to avoid apparently singular contributions arising from the critical layers where G is infinite if ϵ is zero. Equation (16) demonstrates that these singularities are easily coped with, the second term being the usual description of Mach waves which Phillips (1960) showed to originate from those critical layers. But the foregoing analysis is equally valid when ϵ is zero everywhere, as we shall now assume it to be, so that T_+ can be described more compactly and in fact recognized to be the field of a moving quadrupole distribution. Once this is done there is no remaining problem with singularities at the critical layer for the theory of convected quadrupoles is sufficiently general to cope quite easily with arbitrary convection speed.

On setting $\epsilon = 0$, equation (16) reduces to

$$T_+ = \frac{x_i x_j}{|\mathbf{x}|^2 c_0^2} \int_V \left[\rho \frac{D}{Dt}\left\{\frac{1}{(1-M_r)} \frac{D}{Dt}\left(\frac{T_{ij}}{\rho(1-M_r)}\right)\right\}\right] d^3\mathbf{y}. \tag{17}$$

This is actually the field induced by the quadrupole field T_{ij} when the source elements move with the fluid particles. This can be demonstrated by expressing the distribution in Lagrangian co-ordinates $(\boldsymbol{\eta}, \tau)$ which are related to the Eulerian system $(\mathbf{y}, \tau)$ through the local velocity. When $\mathbf{y}$ locates a specific fluid particle it is a function of $\boldsymbol{\eta}$ and τ determined from the equation

$$\partial y_i(\boldsymbol{\eta}, \tau)/\partial \tau = u_i. \tag{18}$$

We write the Lagrangian description of the quadrupole field as

$$T_{ij}(\mathbf{y}, \tau) = q_{ij}(\boldsymbol{\eta}, \tau). \tag{19}$$

The induced field is given in equation (3.6) of Ffowcs Williams & Hawkings (1969):

$$4\pi c_0^2(\rho - \rho_0) = \frac{\partial^2}{\partial x_i \partial x_j} \int q_{ij}(\boldsymbol{\eta}, \tau)\, \delta\left(\tau - t + \frac{r}{c_0}\right) \frac{J}{r}\, d^3\boldsymbol{\eta}\, d\tau, \tag{20}$$

where $r = |\mathbf{x} - \mathbf{y}(\boldsymbol{\eta}, \tau)|$ and J is the Jacobian of the transformation from fixed to moving axes. Since the axes move with the fluid particles in our case, volume elements in the two co-ordinate systems are related through the ratio of the density of a fluid particle to the density $\rho^*(\mathbf{y}) = \rho^*(\boldsymbol{\eta})$ of that particle at the time when the two co-ordinate systems coincide. The continuity equation can be used to evaluate J according to equation (3.8) of Ffowcs Williams & Hawkings' paper to give

$$J = \rho^*/\rho, \quad \text{i.e.} \quad \rho^* d^3\boldsymbol{\eta} = \rho\, d^3\mathbf{y}. \tag{21}$$

In the far field the gradient operator acts only through the retarded time, so that

$$\frac{\partial}{\partial x_i} \sim \frac{-1}{c_0} \frac{x_i}{|\mathbf{x}|} \frac{\partial}{\partial t}, \tag{22}$$

and (20) becomes asymptotically as $|\mathbf{x}| \to \infty$

$$4\pi c_0^4(\rho - \rho_0) = \frac{x_i x_j}{|\mathbf{x}|^3} \int q_{ij}(\boldsymbol{\eta}, \tau) \frac{\partial^2}{\partial t^2} \delta\left(\tau - t + \frac{r}{c_0}\right) \frac{\rho^*}{\rho}\, d^3\boldsymbol{\eta}\, d\tau. \tag{23}$$

Now,
$$\frac{\partial r}{\partial \tau} = \frac{\partial}{\partial \tau} |\mathbf{x} - \mathbf{y}(\boldsymbol{\eta}, \tau)| = \frac{-(x_i - y_i)}{|\mathbf{x} - \mathbf{y}|} u_i = -c_0 M_r, \tag{24}$$

so that
$$\frac{\partial}{\partial t} \delta\left(\tau - t + \frac{r}{c_0}\right) = \frac{-1}{(1-M_r)} \frac{\partial}{\partial \tau} \delta\left(\tau - t + \frac{r}{c_0}\right),$$

and (23) can be written as

$$4\pi c_0^4(\rho - \rho_0) = \frac{x_i x_j}{|\mathbf{x}|^3} \int \frac{\rho^*}{\rho} q_{ij}(\boldsymbol{\eta}, \tau) \frac{1}{(1-M_r)} \frac{\partial}{\partial \tau}\left\{\frac{1}{(1-M_r)} \frac{\partial}{\partial \tau} \delta\left(\tau - t + \frac{r}{c_0}\right)\right\} d^3\boldsymbol{\eta}\, d\tau. \tag{25}$$

This equation is now twice integrated by parts with respect to time,

$$4\pi c_0^4(\rho-\rho_0) = \frac{x_i x_j}{|\mathbf{x}|^3}\int \frac{\partial}{\partial \tau}\left\{\frac{1}{(1-M_r)}\frac{\partial}{\partial \tau}\left(\frac{q_{ij}}{\rho(1-M_r)}\right)\right\}\rho^*\delta\left(\tau-t+\frac{r}{c_0}\right)d^3\eta\, d\tau, \tag{26}$$

and re-expressed in its Eulerian form, to give

$$4\pi c_0^2|\mathbf{x}|(\rho-\rho_0) = \frac{x_i x_j}{|\mathbf{x}|^2 c_0^2}\int\left[\rho\frac{D}{Dt}\left\{\frac{1}{(1-M_r)}\frac{D}{Dt}\left(\frac{T_{ij}}{\rho(1-M_r)}\right)\right\}\right]d^3\mathbf{y}, \tag{27}$$

which is seen to be identical with (17). The direct radiation field T_+ is therefore expressible in several forms, the most compact of which is

$$T_+ = \frac{\partial^2}{\partial x_i \partial x_j}\int\left[\frac{\rho^*}{\rho}q_{ij}\frac{d^3\eta}{(1-M_r)}\right]. \tag{28}$$

This is the field that would be radiated into an unbounded medium at rest by a quadrupole distribution T_{ij} moving with the same velocity field as the fluid itself.

The pressure and velocity perturbations on S are related through the equations governing small amplitude motion about the mean state in the excluded region. That relation allows (15) to be solved, the general procedure being to recognize and exploit a further analogy.

On S, since $l_i u_i$ is a small perturbation about the uniform mean velocity that lies parallel to S, it can be expressed in terms of $\xi(\mathbf{y},t)$, the normal displacement of a fluid particle from its rest position on S. Now if we assume, without any loss of generality, that the uniform stream moves in the x_1 direction with speed U_1, then

$$l_i u_i = \partial\xi/\partial t + U_1\,\partial\xi/\partial x_1 \tag{29}$$

and we can express the surface velocity term in (15) as a function of the particle displacement:

$$\frac{\rho_0}{(1-M_r)}\frac{\partial}{\partial t}\int_S l_i[u_i]\,d^2\mathbf{y} = \frac{\rho_0}{(1-M_r)}\frac{\partial}{\partial t}\int_S\left[\frac{\partial\xi}{\partial t}+U_1\frac{\partial\xi}{\partial y_1}\right]d^2\mathbf{y}. \tag{30}$$

But,

$$\int_S\left[U_1\frac{\partial\xi}{\partial y_1}\right]d^2\mathbf{y} = U_1\int\left\{\frac{\partial}{\partial y_1}[\xi]+\frac{\partial}{\partial x_1}[\xi]\right\}d^2\mathbf{y}, \tag{31}$$

and as the first term integrates directly to zero and the second term becomes, in the far field,

$$-\frac{U_1}{c_0}\frac{x_1}{|\mathbf{x}|}\frac{\partial}{\partial t}\int_S[\xi]\,d^2\mathbf{y} = -M_r\frac{\partial}{\partial t}\int_S[\xi]\,d^2\mathbf{y}, \tag{32}$$

equation (30) can be recognized as a familiar term (cf. Gottlieb 1960) in the theory of sound generation near vortex sheets:

$$\frac{\rho_0}{(1-M_r)}\frac{\partial}{\partial t}\int_S l_i[u_i]\,d^2\mathbf{y} = \int_S\left[\rho_0\frac{\partial^2\xi}{\partial t^2}\right]d^2\mathbf{y}. \tag{33}$$

The field radiated to large distances is consequently identical with that in an analogous vortex-sheet problem, (15) now giving its strength as

$$4\pi c_0^2\,|\mathbf{x}|\,H(\rho-\rho_0)(\mathbf{x},t) = T_+ - \frac{x_i}{|\mathbf{x}|}\frac{1}{c_0}\frac{\partial}{\partial t}\int_S l_i[p]\,d^2\mathbf{y} - \int_S\left[\rho_0\frac{\partial^2\xi}{\partial t^2}\right]d^2\mathbf{y}, \tag{34}$$

with T_+ given by (16), or equivalently (17) or (28).

The field in the uniformly moving fluid beyond the surface S is a linear sound field where p is determined on S once ξ is specified together with a radiation or finiteness condition. Further, since the field radiated into the static fluid is determined once p is a known function of ξ, the analogy is clear. An example illustrating the principle that the inter-relation between surface terms determines the field is given by Ffowcs Williams (1965*a*).

We have then proved that the sound field radiated by turbulence formed at the edge of a stream mixing with its environment is identical with that generated in a much simpler model problem. This is obvious when it is appreciated that (34) *also* represents the field that would be radiated to the particular distant point $\mathbf{x}$ by the quadrupole field T_{ij} were it being convected with the actual fluid particles but radiating into homogeneous fluid at rest in V. In the analogy, that fluid is bounded by a vortex sheet at S beyond which there is a weakly perturbed fluid motion, the particle displacement and pressure being held continuous across S. The dependence of p on ξ, and not their individual values, determines the sound. That dependence is imposed entirely by the uniformly moving stream beyond S in both the real and analogous vortex-sheet problems.

Given a description of the turbulence, and hence T_+, the integral equation (34) can in general be solved through this analogy with the vortex-sheet problem, and for that solution the vortex-sheet instabilities are *irrelevant* because the problem we have treated exactly always has a bounded solution. The analogy does *not* describe the field in the vicinity of the vortex sheet, and so the detailed local source conditions, which together with causality constraints on an actual vortex sheet usually require exponential growth, are quite different in the real and analogous problems. The real problem calls for compliance with a distant radiation condition both within and exterior to the moving fluid. It also requires finiteness everywhere and as Jones & Morgan (1972) have pointed out, local causality must then be violated in the model problem. But this is of no concern in our analogy, which is quite incapable of interpretation within the source region, where the physical contradiction would occur were the problem actually one of an excited unstable vortex sheet.

The analogy between the actual field and that generated by the convected quadrupoles interacting with a vortex sheet is exact *irrespective of the location of that sheet* provided that it lies in and parallel to the linearly disturbed, otherwise uniformly moving fluid. There is clearly some scope for ambiguous interpretation of this result, a point that again serves to emphasize the major role played by linear elements of Lighthill's stress tensor whenever they are distributed over an extensive field. By placing the surface deep inside the moving fluid, the source field has such an extensive linear distribution, which must represent the difference between the response of a nearby and a distant vortex sheet. The interpretation is straightforward only when the surface can be positioned close enough to the real source field that an insufficient volume of linearly disturbed moving fluid exists within V to contribute any significant effect. That is the case when the surface is positioned at a distance from the source small compared with a wavelength. Then, too, phase differences between those directly radiating elements and those represented by the vortex-sheet response are negligible and the field can be evaluated without paying close attention to variations of retarded time. This analogy is therefore expected to be most useful at

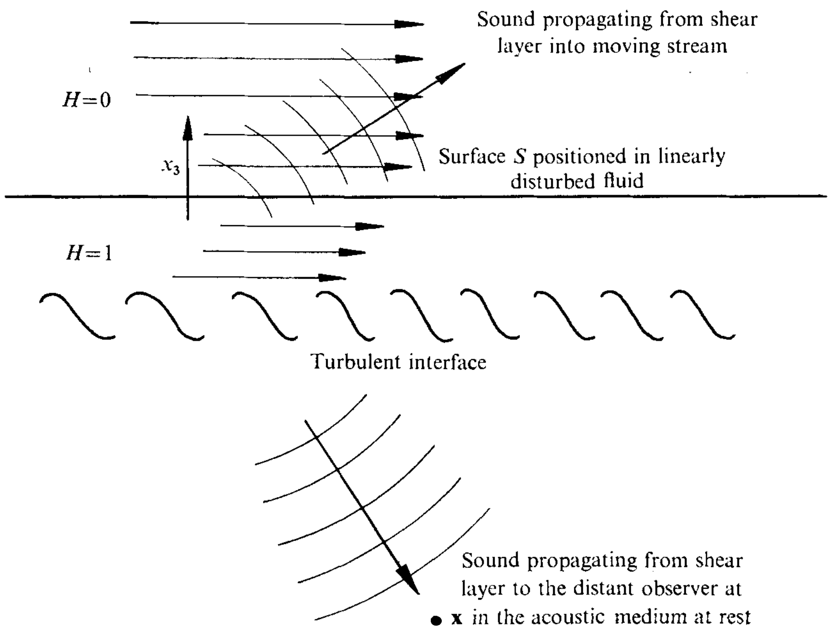


FIGURE 2. Illustration of the geometry in the special problem of a plane turbulent shear layer.

low Mach numbers when the acoustic wavelength is long on the source scale, though at higher Mach numbers, just like Lighthill's analogy, on which it rests, it remains formally exact.

2. The plane shear layer

The scheme of solution can be simplified in the particular case we now treat because we choose a straightforward geometry. The plane S is the surface

$$x_3 = 0, \quad l_3 = 1, \quad l_1 = l_2 = 0.$$

The fluid of density ρ_1 in $x_3 > 0$ moves in the x_1 direction with uniform speed U_1 apart from linear perturbations driven from $x_3 < 0$; see figure 2.

The velocity and pressure integrals in (15) can be related very simply if we represent the field variables by their space-time Fourier transforms, viz.,

$$\int_S [p]\, d^2\mathbf{y} = \int_S \left[\iint p(y_3, k, \omega)\, e^{i(\mathbf{k}.\mathbf{y}+\omega t)} d^2\mathbf{k}\, d\omega\right] d^2\mathbf{y} \tag{35}$$

$$= \int_S \iint p(y_3, k, \omega)\, e^{i\mathbf{k}.\mathbf{y}} \exp i\omega\left(t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x}.\mathbf{y}}{|\mathbf{x}|\, c_0}\right) d^2\mathbf{k}\, d\omega\, d^2\mathbf{y}. \tag{36}$$

The asymptotic value of the retarded time has been chosen; it has the required degree of accuracy in the distant radiation field. The $\mathbf{y}$ surface integration yields a delta function that allows the $\mathbf{k}$ surface integral to be evaluated trivially, so that

$$\int [p]\, d^2\mathbf{y} = \int_{-\infty}^{\infty} p\left(y_3, -\frac{\omega \mathbf{x}}{c_0|\mathbf{x}|}, \omega\right) \exp\{i\omega(t - |\mathbf{x}|/c_0)\}\, d\omega. \tag{37}$$

Now $p(y_3, \mathbf{k}, \omega)$ satisfies the convective form of the Helmholtz equation in $y_3 > 0$:

$$\{\partial/\partial t + c_0 M \partial/\partial x_1\}^2 p - c_1^2 \nabla^2 p = 0, \quad c_0 M = U_1; \tag{38}$$

$$\{c_1^2 \partial^2/\partial y_3^2 + (\omega + c_0 M k_1)^2 - c_1^2(k_1^2 + k_2^2)\} p(y_3, \mathbf{k}, \omega) = 0, \tag{39}$$

$$p(y_3, \mathbf{k}, \omega) = p(0, \mathbf{k}, \omega) \exp\{-i((\omega + c_0 M k_1)^2 - c_1^2(k_1^2 + k_2^2))^{\frac{1}{2}} y_3/c_1\} \tag{40}$$

is the solution complying with the radiation condition, so that

$$c_1 \partial p(y_3, \mathbf{k}, \omega)/\partial y_3 = -i\{(\omega + c_0 M k_1)^2 - c_1^2(k_1^2 + k_2^2)\}^{\frac{1}{2}} p(y_3, \mathbf{k}, \omega). \tag{41}$$

The linearized momentum equation in the uniform flow is

$$\frac{\partial u_3}{\partial t} + c_0 M \frac{\partial u_3}{\partial y_1} + \frac{1}{\rho_1} \frac{\partial p}{\partial y_3} = 0,$$

or $$\rho_1 c_1 i(\omega + c_0 M k_1)\, u_3(y_3, \mathbf{k}, \omega) = i\{(\omega + c_0 M k_1)^2 - c_1^2(k_1^2 + k_2^2)\}^{\frac{1}{2}} p(y_3, \mathbf{k}, \omega). \tag{42}$$

Consequently,

$$p\left(y_3, -\frac{\omega \mathbf{x}}{c_0|\mathbf{x}|}, \omega\right) = p\left(y_3, -\frac{\omega \mathrm{x}_1}{c_0|\mathbf{x}|}, -\frac{\omega \mathrm{x}_2}{c_0|\mathbf{x}|}, \omega\right)$$

$$= \frac{\rho_1 c_1 (1 - \bar{M}_r)\, u_3\left(y_3, -\dfrac{\omega}{c_0}\dfrac{\mathbf{x}}{|\mathbf{x}|}, \omega\right)}{\left\{(1 - \bar{M}_r)^2 - \dfrac{c_1^2}{c_0^2}\dfrac{(x_1^2 + x_2^2)}{|\mathbf{x}|^2}\right\}^{\frac{1}{2}}}, \tag{43}$$

where $\bar{M}_r = M x_1/|\mathbf{x}|$, and (37) can be written to relate the pressure and velocity terms in a specific way:

$$\int_S [p]\, d^2\mathbf{y} = \frac{\rho_1 c_1 (1 - \bar{M}_r)}{\left\{(1 - \bar{M}_r)^2 - \dfrac{c_1^2}{c_0^2}\dfrac{(x_1^2 + x_2^2)}{|\mathbf{x}|^2}\right\}^{\frac{1}{2}}} \int_S [u_3]\, d^2\mathbf{y}. \tag{44}$$

If the radical is imaginary, (44) is of course not valid and (43) must be used instead. But the imaginary number is simply an indication that the pressure and velocity terms are 90° out of phase and as long as they are treated accordingly, we may continue to use (44), which has the advantage of giving a 'broad band' result but the disadvantage that it is then impossible to specify the 90° phase shift in a straightforward manner. With this expression (15) is solved. When the observation point $\mathbf{x}$ is in the still fluid the equation amounts to

$$4\pi c_0^2 |\mathbf{x}| (\rho - \rho_0)(\mathbf{x}, t)$$

$$= T_+ - \frac{\rho_0}{(1 - \bar{M}_r)} \frac{\partial}{\partial t} \int_S [u_3]\, d^2\mathbf{y} \left\{1 + \frac{x_3}{|\mathbf{x}|} \frac{(1 - \bar{M}_r)^2}{\rho_0 c_0} \frac{\rho_1 c_1}{\left\{(1 - \bar{M}_r)^2 - \dfrac{c_1^2}{c_0^2}\dfrac{(x_1^2 + x_2^2)}{|\mathbf{x}|^2}\right\}^{\frac{1}{2}}}\right\} \tag{45}$$

and when $\mathbf{x}$ is outside V at the position of the specular image in the plane $x_3 = 0$, i.e. when $\mathbf{x} = (x_1, x_2, -x_3)$, then (15) is

$$0 = T_- - \frac{\rho_0}{(1 - \bar{M}_r)} \frac{\partial}{\partial t} \int_S [u_3]\, d^2\mathbf{y} \left\{1 - \frac{x_3}{|\mathbf{x}|} \frac{(1 - \bar{M}_r)^2}{\rho_0 c_0} \frac{\rho_1 c_1}{\left\{(1 - \bar{M}_r)^2 - \dfrac{c_1^2}{c_0^2}\dfrac{(x_1^2 + x_2^2)}{|\mathbf{x}|^2}\right\}^{\frac{1}{2}}}\right\}, \tag{46}$$

where T_- is written for the field that would be radiated to $(x_1, x_2, -x_3)$ by the convected quadrupoles in V were they to exist in an unbounded homogeneous fluid at rest. The linear surface term can then be eliminated to give

$$(\rho-\rho_0)(\mathbf{x},t) = \frac{1}{4\pi c_0^2|\mathbf{x}|}\{T_+ + RT_-\}, \tag{47}$$

where

$$R = \left\{\frac{\dfrac{x^3}{|\mathbf{x}|}(1-\bar{M}_r)^2\dfrac{\rho_1}{\rho_0}+\left\{\dfrac{c_0^2}{c_1^2}(1-\bar{M}_r)^2-\dfrac{(x_1^2+x_2^2)}{|\mathbf{x}|^2}\right\}^{\frac{1}{2}}}{\dfrac{x_3}{|\mathbf{x}|}(1-\bar{M}_r)^2\dfrac{\rho_1}{\rho_0}-\left\{\dfrac{c_0^2}{c_1^2}(1-\bar{M}_r)^2-\dfrac{(x_1^2+x_2^2)}{|\mathbf{x}|^2}\right\}^{\frac{1}{2}}}\right\}. \tag{48}$$

Thus, as we have already stated, the effect of the mean flow and density change is compactly represented in this equivalence of the actual field with that which would be radiated by the convected quadrupoles were they adjacent to a plane of discontinuity at $x_3 = 0$ separating the $\rho_0 c_0$ still fluid from the moving $\rho_1 c_1$ fluid. R, the reflexion coefficient for that problem, was found by Miles (1956) to be that given above.

Parametric form of the radiated field

The mean flow interference effects are only straightforward in certain particular cases, one of which being the long-wave limit where the turbulent interface between the moving and still fluid is thin on the wavelength scale. Then the image sources can be assumed, when viewed from afar, coincident with the real sources and the phase change arising from their small separation neglected.

The magnitude of the radiation field is then obtained very simply from (47), being

$$\overline{(\rho-\rho_0)^2}(\mathbf{x}) = \frac{\overline{T_+^2}}{16\pi^2 c_0^4|\mathbf{x}|^2}|1+R|^2 f(\mathbf{x}/|\mathbf{x}|) \tag{49}$$

for longitudinal quadrupoles (T_{11}, T_{22} and T_{33}) and the (1, 2) lateral quadrupoles, where the field is symmetric about any $x_3 = 0$ quadrupole axis, and

$$\overline{(\rho-\rho_0)^2}(\mathbf{x}) = \frac{\overline{T_+^2}}{16\pi^2 c_0^4|\mathbf{x}|^2}|1-R|^2\frac{x_3^2}{|\mathbf{x}|^4}(x_1^2 \text{ or } x_2^2), \tag{50}$$

depending on the type of quadrupole, for the field of the lateral quadrupoles, T_{13}, T_{23}, etc., which are antisymmetric about their $x_3 = 0$ axes, their specular images then opposing the direct field. $f(\mathbf{x}/|\mathbf{x}|)$ is a product of direction cosines which we set equal to unity in what follows. These expressions hold whether R is real or complex. The phase quadrature effect signified by a complex reflexion coefficient is properly accounted for.

The most striking single feature of this theory is the impossibility of aerodynamic sound propagation parallel to the plane of the mixing layer. There $x^3/|\mathbf{x}|$ is zero, and R is -1. Only longitudinal quadrupoles radiate in the downstream direction and their field is annihilated by that of their images. Qualitatively the streamwise zone of silence so produced might be thought of as a refractive effect. However, it emerges in this long-wavelength model as a destructive interference between two terms, one the direct field and the other that scattered off the moving medium. The interference is complete in the plane of the vortex sheet.

Whenever the turbulence level is a small fraction of the local mean velocity then to first order the quadrupoles can be assumed to be in uniform motion. One of the least restrictive derivations of the parametric field dependence for that situation is given by Ffowcs Williams (1969); provided that the various turbulence correlations can be scaled into an isotropic form, then

$$\overline{T_+^2} = K(\rho_0+\rho_1)^2 U_1^8 A D^{-5}, \tag{51}$$

where K is a constant and D is the generalized Doppler factor:

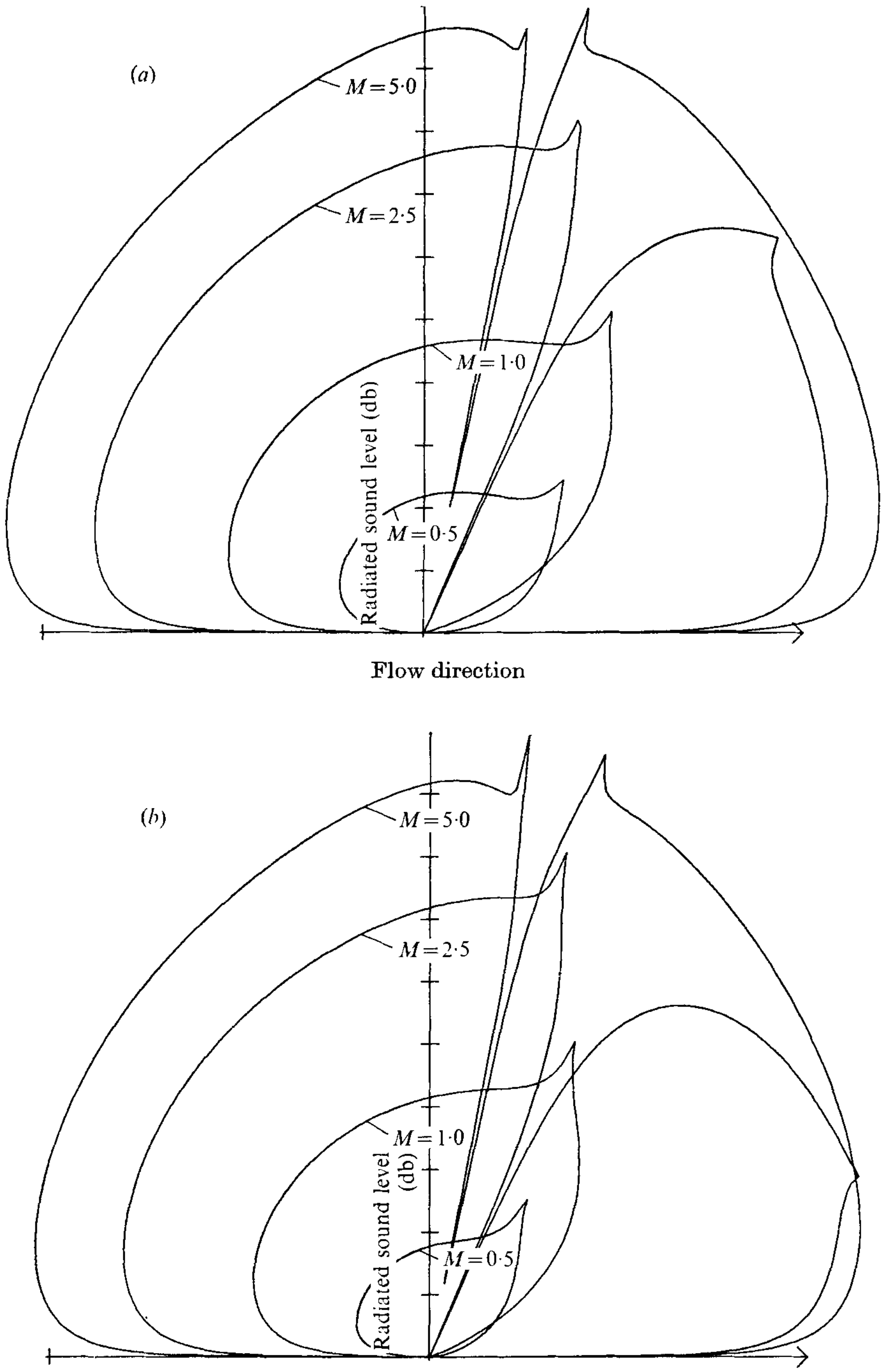
$$D = \{(1-0{\cdot}7\bar{M}_r)^2+\alpha^2\bar{M}_r^2+\beta^2(M^2-\bar{M}_r^2)\}^{\frac{1}{2}}. \tag{52}$$

This is the form of the Reynolds-stress-driven field on a shear layer separating still ρ_0 fluid from a stream of density ρ_1 moving at speed $U_1 = c_0 M$. This parametric form depends on the existence of a definite turbulence length scale $A^{\frac{1}{2}}$ and presupposes that the principal shear-layer sources are moving downstream at 0·7 times the speed of the primary jet flow. The constant α is the ratio of the eddy longitudinal length scale to the distance travelled by that eddy during its coherent life (α^2 is probably about 0·3), and β is the transverse eddy scale divided by the distance travelled by the eddy (β^2 is probably of the order of 0·03).

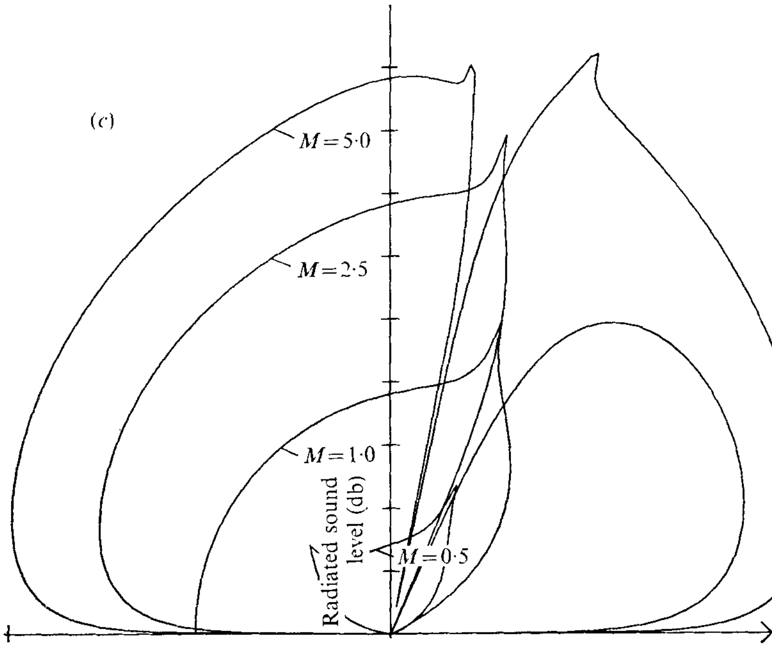
The parametric form of T_+^2 may well be quite different in regions of really intense turbulence where the unsteady elements of $D(1-M_r)^{-1}/Dt$ are as significant as unsteady parts of T_{ij}. Also, when non-isentropic effects are large there is a quite different basic dependence on velocity, the fourth power of velocity taking over from the basic eighth-power law and that in turn being subject to multiplication by the $|1+R|^2$ of (49). But we do not consider those ideas in any detail as we now restrict our study to the straightforward case of relatively low turbulence level when (49)–(51) describe the radiation field. Also we limit our attention to gases with a common ratio of specific heats so that the product ρc^2 is invariant in the constant pressure mean flows.

Figures 3(*a*)–(*c*) show various plots of (49) when $x_2 = 0$, in polar form. The decibel sound level, which is essentially $10\log_{10}$ of equation (49) is plotted as a function of the angle to the flow direction. The scale is referenced to a convenient but arbitrary datum. The three sets of curves depict the directionality and relative magnitude of the field at several distinct values of the mean flow velocity $c_0 M$ and density ratio ρ_0/ρ_1. These curves bear a close similarity to those given by Gottlieb (1960) for the radiation from a fixed point source adjacent to a vortex sheet. In fact it is the properties of the vortex-sheet model that dominate the field as can be seen from figures 4(*a*)–(*c*), which show the various elements of (49) plotted separately and then superimposed. $\overline{T_+^2}$ is plotted first in decibel form, for a mean flow velocity of $2{\cdot}5c_0$ and density of $0{\cdot}5\rho_0$. Then in figure 4(*b*) $|1+R|^2$ is plotted, again in decibel form, for the same flow conditions. Figure 4(*c*) shows the composite field that represents the radiation from convected shear-layer turbulence, one of the curves in figure 3(*b*) being the sum of the decibel values in figures 4(*a*) and (*b*).

Figure 5 is a polar diagram for the field of lateral quadrupoles that are antisymmetric about $x_3 = 0$, which is represented by (50); again x_2 is set equal to zero.

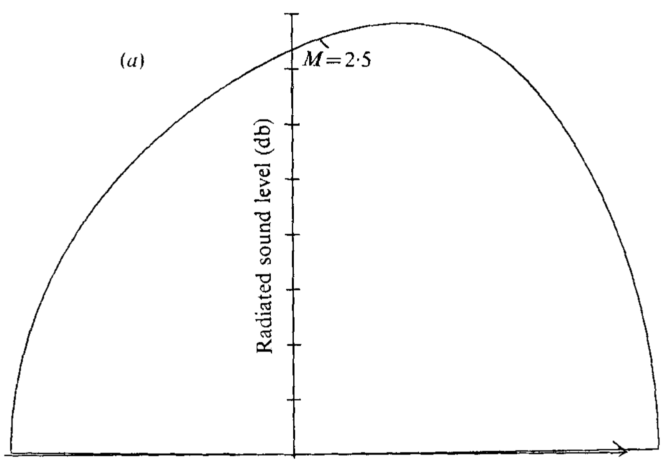


FIGURES 3(*a*, *b*). For legend see next page.



Flow direction

FIGURE 3. Polar diagrams of the sound radiated by turbulence at the edge of a steady stream. (a) $\rho_0/\rho_1 = 1{\cdot}0$. (b) $\rho_0/\rho_1 = 2{\cdot}0$. (c) $\rho_0/\rho_1 = 4{\cdot}0$. On this and similar figures each interval on the ordinate represents 10 db.



Flow direction

FIGURE 4(a). For legend see next page.

These figures all indicate that the presence of the mean flow influences the generation and possibly the propagation of sound in a very dramatic manner. The basic turbulence was assumed devoid of any distinct structure, as was the sound field it would radiate into a homogeneous environment at rest. But the mean flow imposes a structure de-emphasizing somewhat the convection-induced

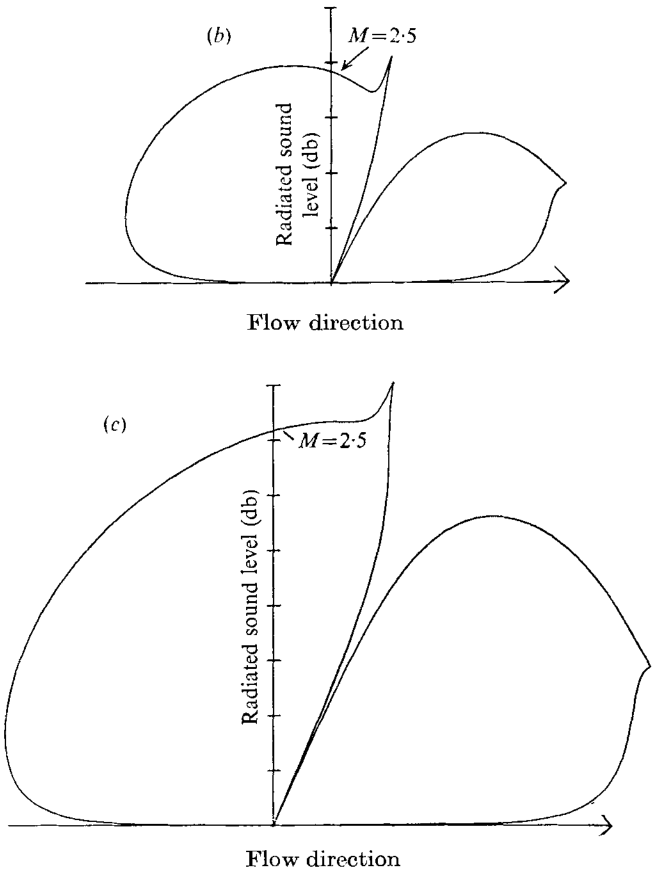


FIGURE 4. Polar diagrams for (a) $\overline{T_+^2}$, (b) $|1+R|^2$ and (c) composite field. $\rho_0/\rho_1 = 2$.

Mach-wave beam and inducing its own directional peaks and troughs, which include the zone of silence in the source plane $x_3 = 0$. The mean flow effects change significantly as the flow velocity is increased, the most dramatic changes being in the rear arc close to the direction of flow. Figures 6 (*a*)–(*e*) show this effect, giving the decibel value of the sound field according to (49) as a function of the flow velocity $c_0 M$ for several angles $\tan^{-1}(-x_3/x_1)$ to the jet direction for various density ratios. Figure 7 shows the way in which the sound varies with jet velocity at 45° to the flow direction when the mean flow interaction effects are neglected. This is given by equation (51). The troughs in the curves at angles within 90° of the flow direction correspond to the condition $R = -1$, which occurs when the phase velocity of the acoustic waves has a component in the direction of flow that is exactly equal to the flow velocity. The flow cannot sustain any such 'static pressure' wave and therefore acts as a pressure release boundary. This effect is evidently capable of negating most of the convective amplification effect. The details depend on the density ratio across the shear layer, and these are depicted in the next figures.

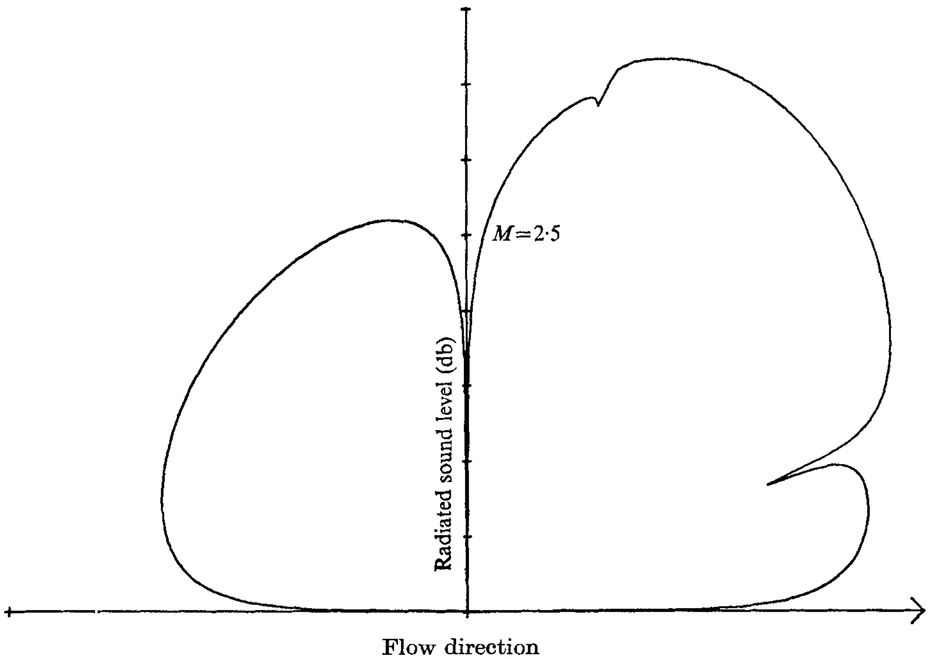


FIGURE 5. Polar diagram of the field of lateral quadrupoles according to (50). $\rho_0/\rho_1 = 2{\cdot}0$.

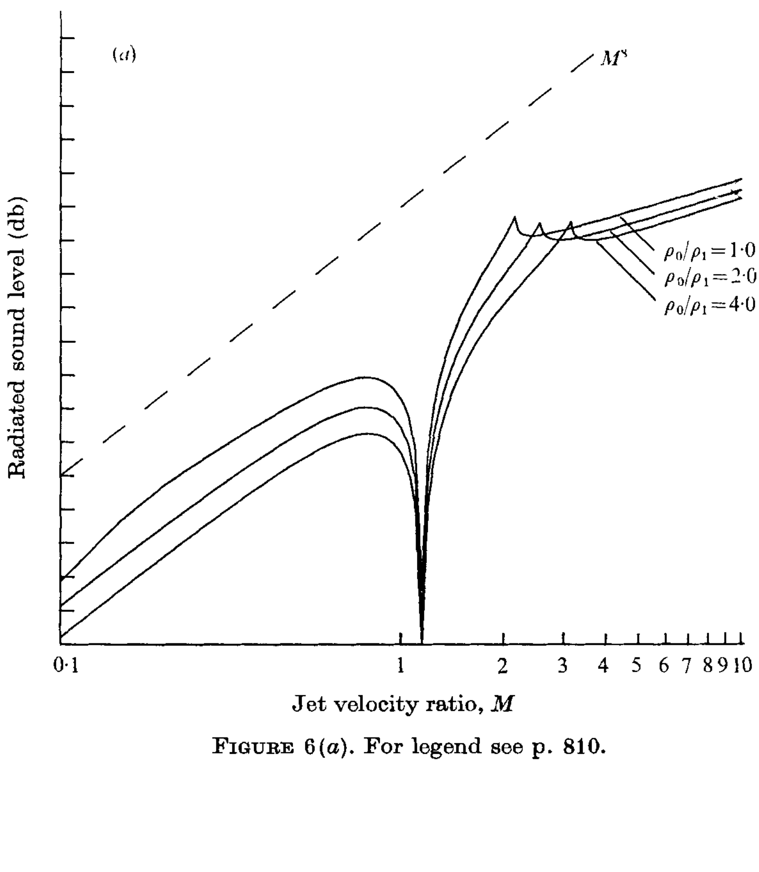
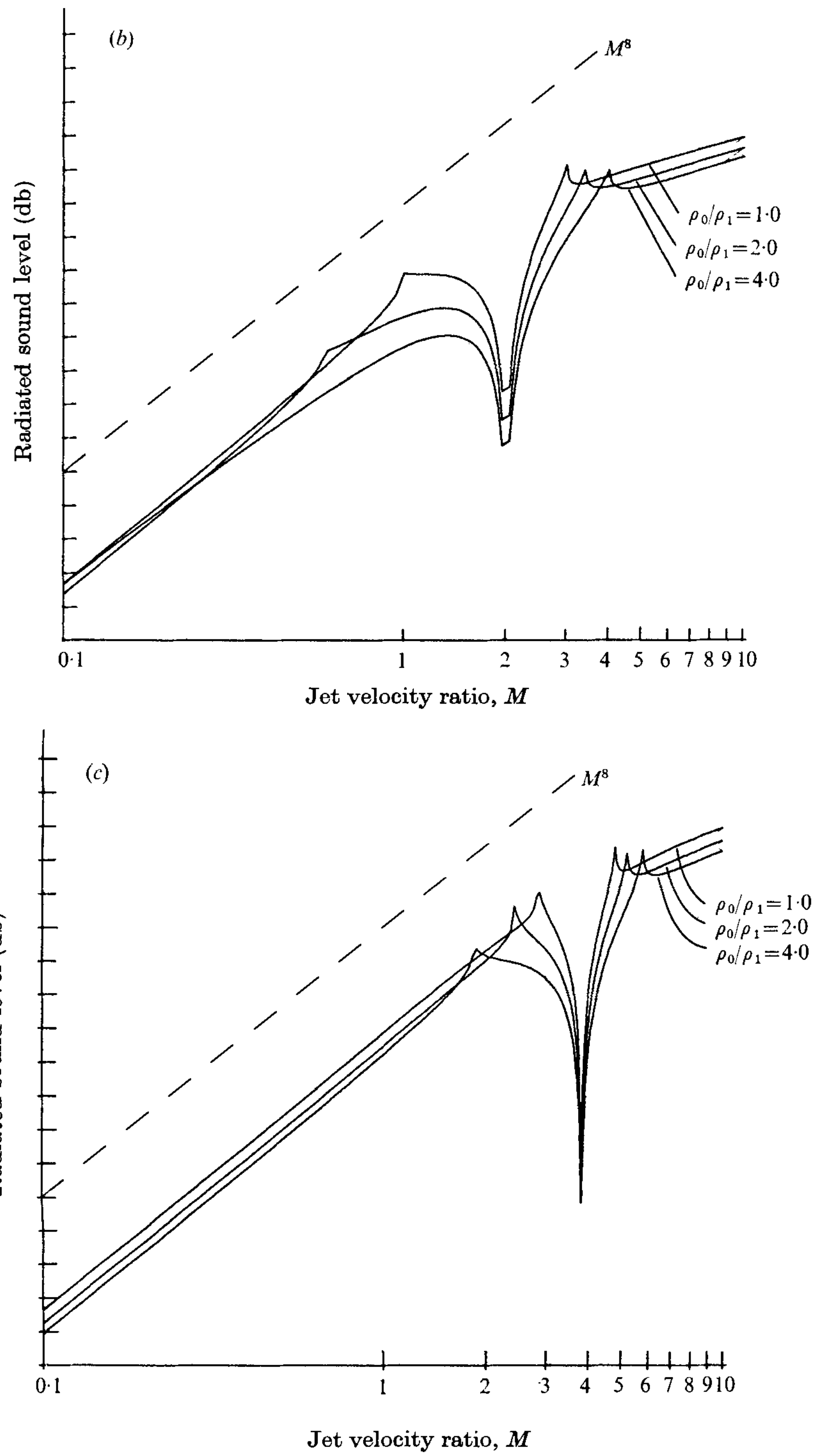
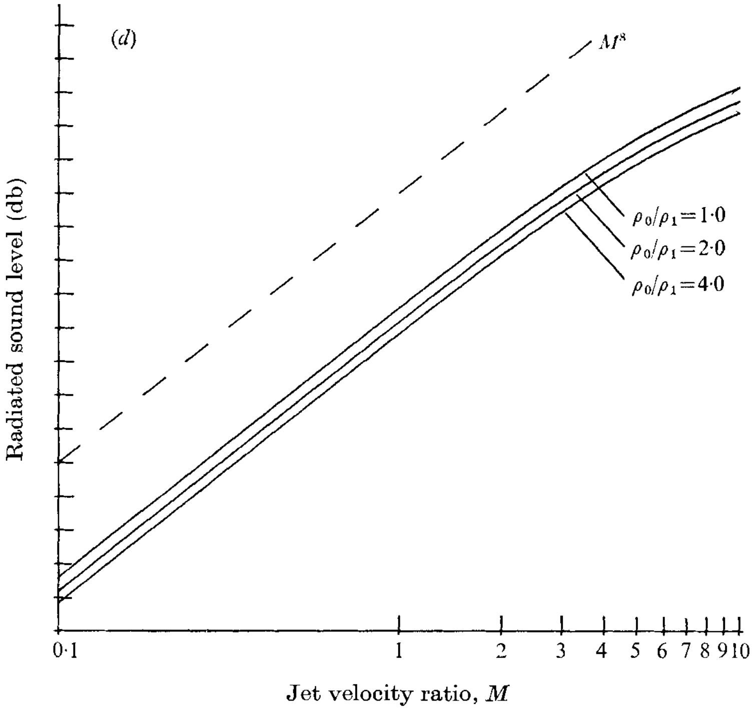


FIGURE 6(*a*). For legend see p. 810.



FIGURES 6(b, c). For legend see next page.



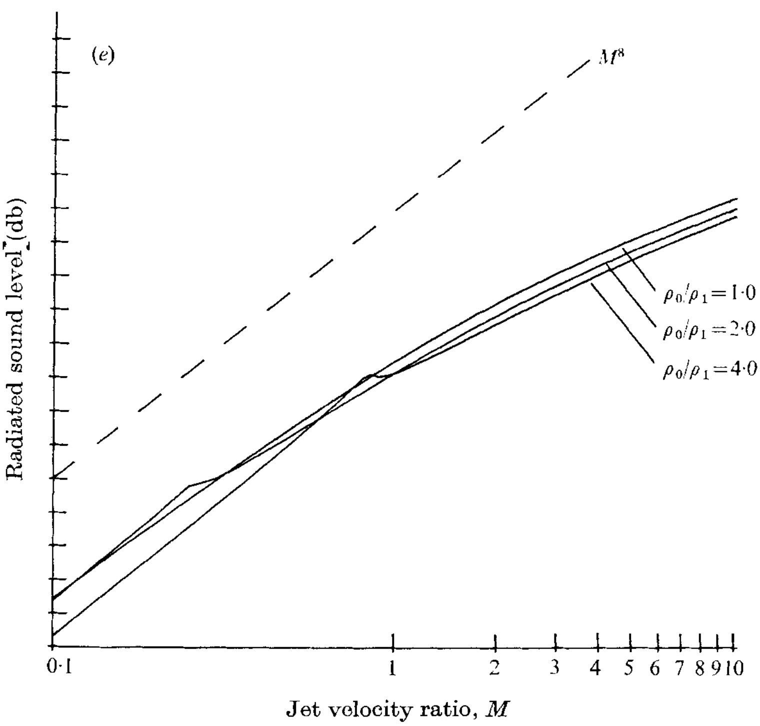


FIGURE 6. Curves showing the sound radiated from turbulence at the edge of a steady stream. $x_2 = 0$. (*a*) $x_1 = -x_3 \cot 30°$. (*b*) $x_1 = -x_3 \cot 60°$. (*c*) $x_1 = -x_3 \cot 75°$. (*d*) $x_1 = 0$. (*e*) $x_1 = -x_3 \cot 150°$.

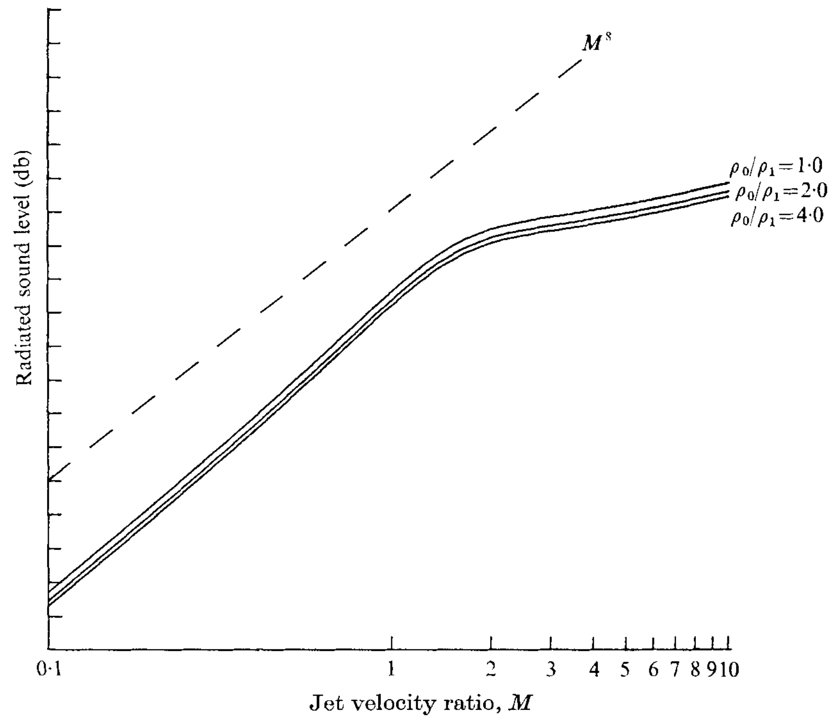


FIGURE 7. Diagram for $\overline{T_+^2}$. $\theta = 45°$.

The variation of sound with density ratio is shown in figures 8(*a*) and (*b*) for three velocity conditions at 30° and 90° to the direction of flow. The variation at other angles differs only in detail. From the figures it is clear that the dependence of the sound on density is not expressible by any simple power law but there is a distinct tendency for the sound to fall more rapidly with flow density at low angles than it does at high angles, particularly when the flow velocity is close to c_0.

Relevance of the model to jet noise

The convective form of Lighthill's acoustic analogy has found widespread application to jet noise problems and is known to explain many of the experimentally observed features in situations of great practical importance (Lighthill 1962). But there are aspects of the jet noise problems which run contrary to that form of the analogy, the most significant being that the high frequency noise does not seem able to propagate at angles close to the jet axis and seems virtually devoid of convective amplification effects. Lush (1971) has published a systematic study of this effect, which has been known for some time and attributed to refractive effects of the mean jet velocity and temperature field, cf. Ribner (1964). Schubert (1969) studied the problem numerically, showing agreement with ray theory at high frequencies and at lower frequencies confirming that flow interaction effects were responsible for the zone of relative silence along the jet axis. His computations showed good agreement with the

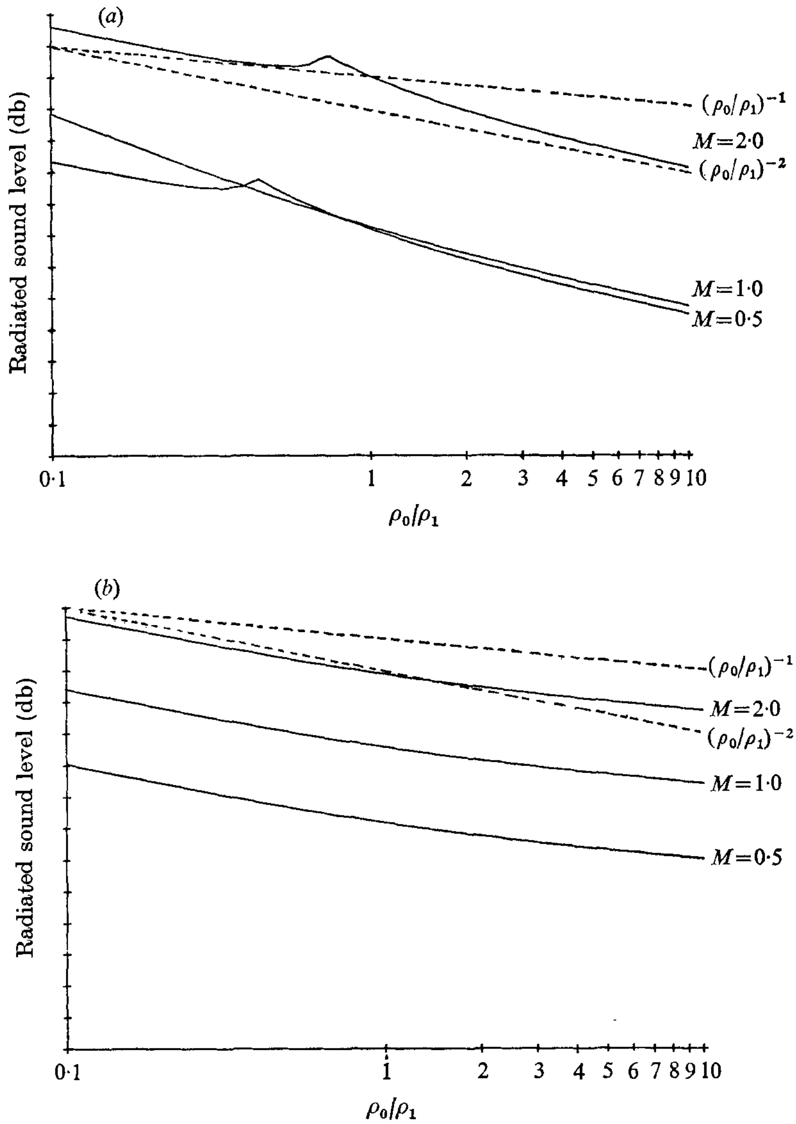


FIGURE 8. Curves showing the sensitivity of sound radiated at the edge of a steady stream to density changes. $x_2 = 0$. (*a*) $x_1 = -x_3 \cot 30°$. (*b*) $x_1 = 0$.

experimental results of Atvars *et al.* (1965) and Grande (1966). It is the high frequency sound that is most affected by this mean flow interaction, sound that radiates in a relatively omnidirectional manner around the jet axis, at least when compared with the predictions of the convective quadrupole model, with a distinct zone of silence close to the jet axis. This current model may well have relevance to that aspect since the high frequency sound is generated by the early thin shear layers bounding a relatively deep uniform-velocity jet core. At least at low Mach numbers, the shear layer must be thin on the wavelength scale, and for sufficiently thin shear layers, the wavelength will be small on the scale of the

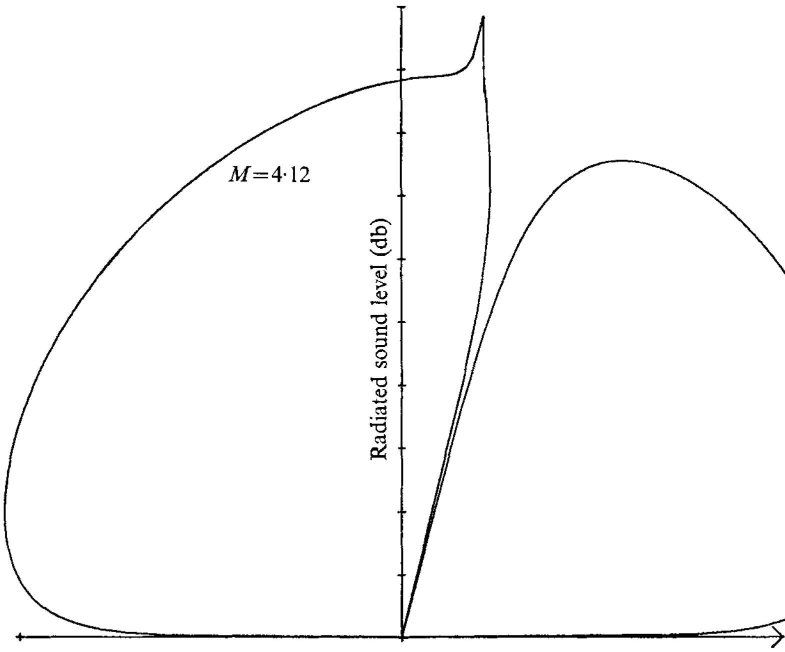


FIGURE 9. Polar diagram of the sound radiated by turbulence at the edge of a steady stream. $\rho_0/\rho_1 = 12{\cdot}2$.

jet diameter. That high frequency sound should therefore be described by the foregoing model, which treats the problem in the opposite limit to that described by ray theory. The model is applicable if the shear layer is sufficiently thin that the wavelength of the sound it produces, though large on the shear-layer scale, is none the less much smaller than the jet diameter. The zone of silence close to the flow direction is certainly consistent with this model, but the strong beams and troughs depicted in the foregoing figures are not generally recognized aspects of the jet noise problem. On the other hand highly directional sound is known to originate from the early shear layers of high-speed jets. Lowson & Ollerhead (1968) published shadowgraph pictures of a strong beam of sound and Tam (1971), by analysing the motion of the developing shear layer, showed the waves to be a characteristic feature of the single interface between the environment and a uniform jet stream. That beam should therefore be a feature of our model also. Figure 9 shows the sound level as a function of the angle to the flow direction in a polar diagram of the values predicted by (49) for the jet conditions illustrated in figure 7 of Tam's paper. Tam computed the beam angle to be 56° to the jet axis and this compared well with the 54° measured in the experiment. Our model also predicts a beam at the observed value, but in addition there is the prediction of a stronger beam at about 80° to the jet axis, right at the boundary of the zone of silence according to ray theory, and that is not evident in the experiment.

Figure 10 (plate 1) reproduces a photograph taken by Westley at the National

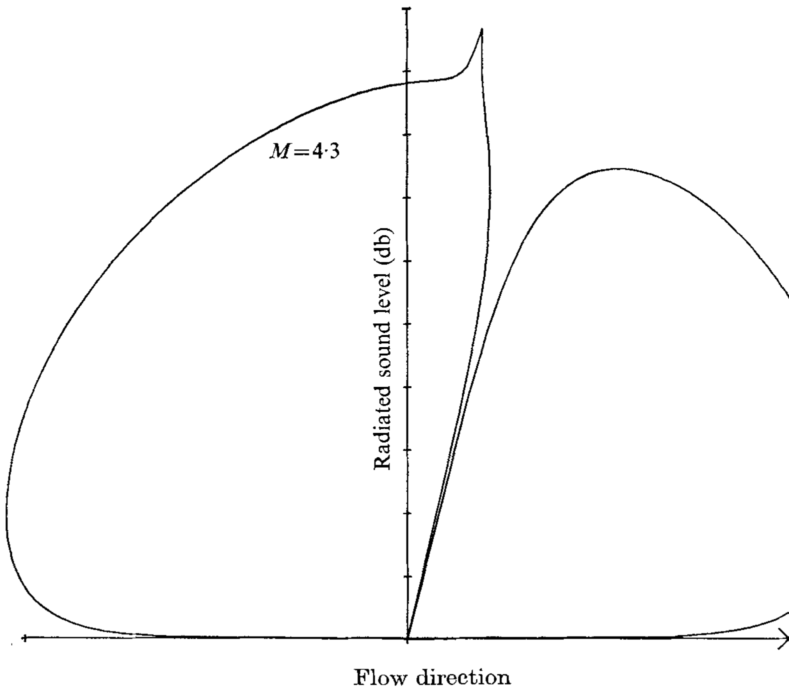


FIGURE 11. Polar diagram of the sound radiated by turbulence at the edge of a steady stream. $\rho_0/\rho_0 = 16{\cdot}9$.

Research Council in Canada with a helium jet discharging into air at a pressure ratio of 4. The jet is small, having a nozzle diameter of 0·04 in., and the discreet wave system it generates is probably unique to flows of relatively low Reynolds number of this type. Figure 11 is the polar diagram of a single-shear-layer aerodynamic source according to (49) at the flow conditions of Westley's jet. Again there is a broad beam at about the angle of that observed experimentally, but again too there is a stronger beam, at the edge of the zone of silence. This is not observed.

These conditions are ones where there is no variation of the sound field within the flow with distance from the interface layer, and there may be good reasons then why our single-layer model fails to explain the field of a double-layer jet. It might do so if there were some artificial stimulus to ensure a coherent interior field. An upstream acoustic source might do that, though of course the directly transmitted sound would mask to some extent the sound created by the shear-layer eddies.

Such an experiment has been performed using the shallow-water analogy where the sound is modelled by surface waves. Figure 12 (plate 2) is a photograph taken at a mean flow speed of $0{\cdot}9c_0$ with upstream acoustic 'seeding' of the jet motion. This experiment illustrates the beam in the vicinity of the angle predicted by this model, the conditions being very close to the $M = 1{\cdot}0$ case shown in figure 3 (a). The beam is also at the angle bordering the zone of silence according to ray theory, so that one cannot determine from the photograph whether the beam is a simple ray or the emission generated at the shear layer provoked by the upstream source.

In conclusion therefore the model problem described in this paper might be quite pertinent to the high frequency sound waves generated by the thin initial jet mixing layers, but some of the strong directional features predicted by the model are yet to be confirmed by direct experiment. The analysis is easily extended to account for a double shear layer and that should improve the relevance of the model in jet noise applications. That extension is now nearing completion by Dr R. Dash.

This work arose out of a need to justify a procedure by which Dr R. Dash and myself had previously modelled mean flow effects on jet noise through vortex-sheet problems. That work will be reported in detail by Dr Dash. I am indebted to Mrs J. Broadway for conducting the numerical computations, to Sir James Lighthill for many helpful comments and Sir Stanley Hooker for stimulating a more penetrating analysis of the high-speed jet noise problem.

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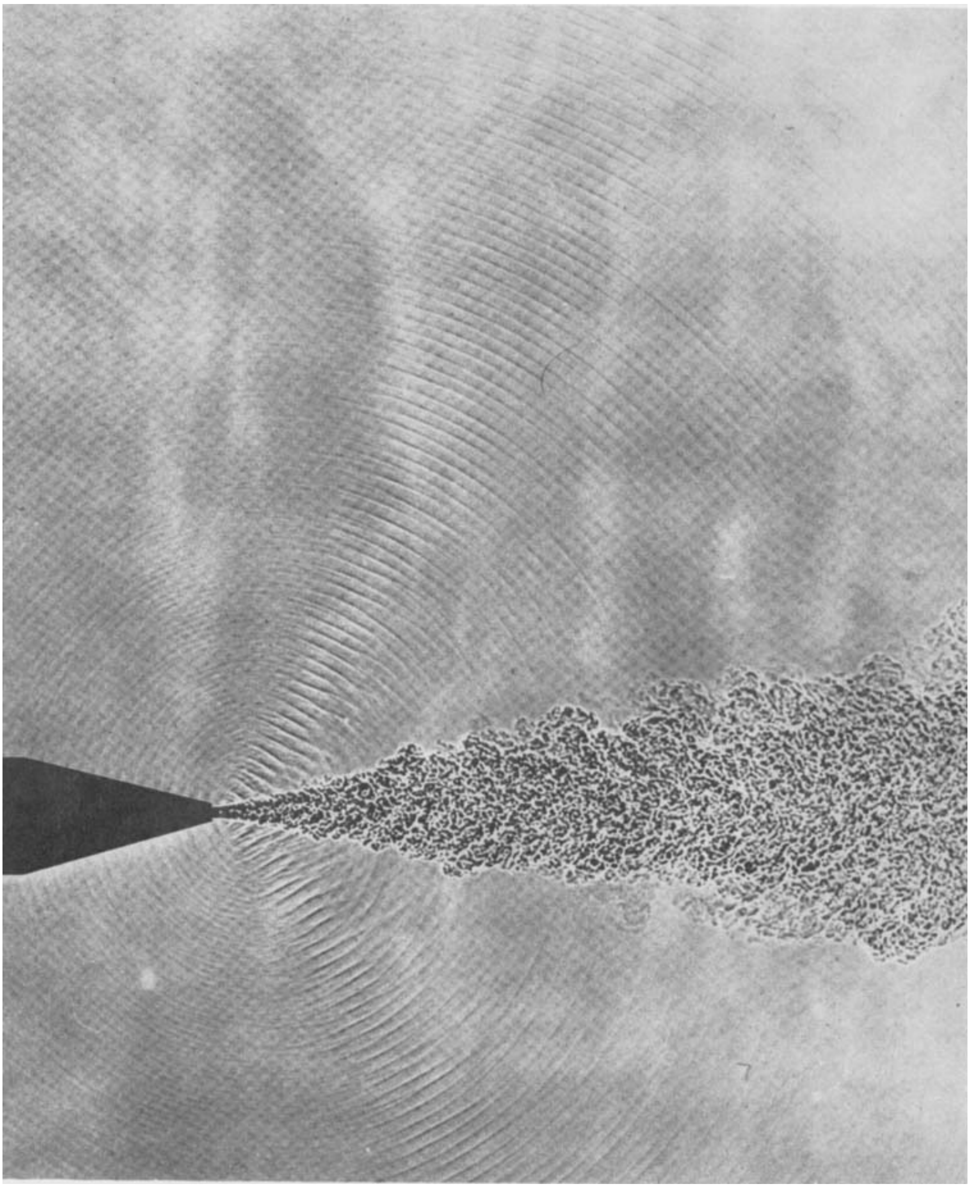

FIGURE 10. Photograph of highly directional waves taken by Westley with a 0·04 in. diameter helium jet at a pressure ratio of 4.

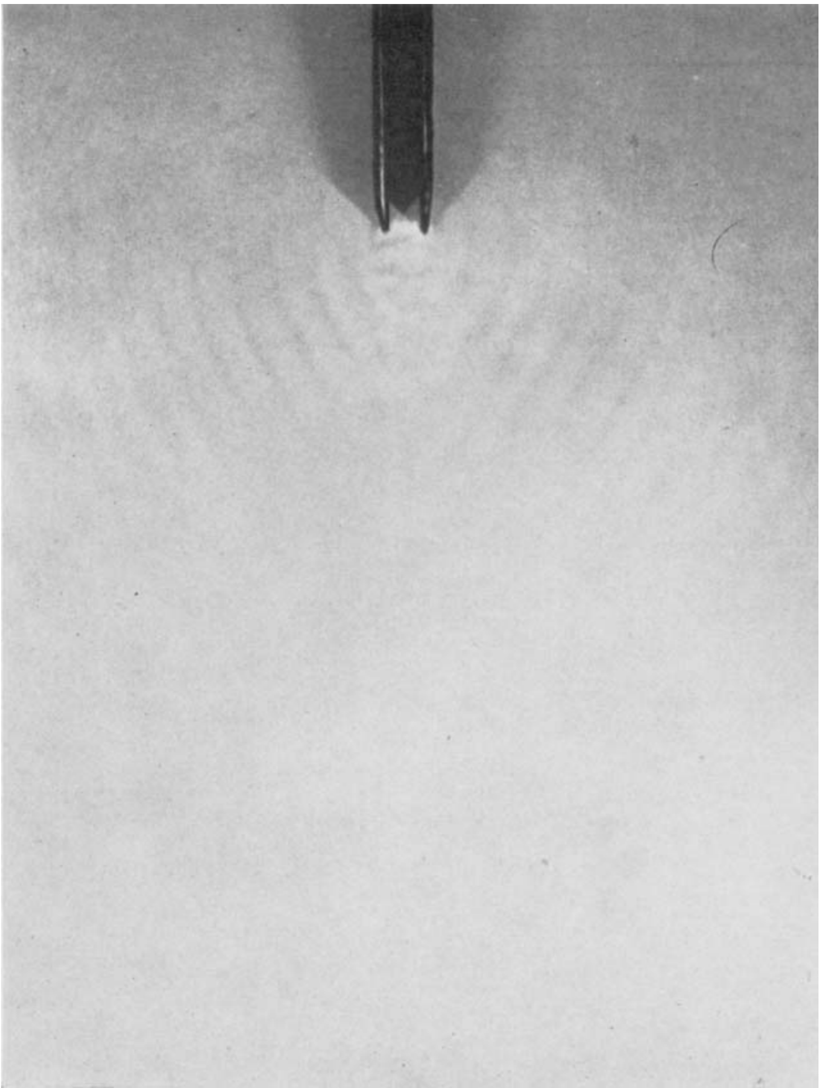

FIGURE 12. Shallow-water simulation of jet noise, showing a highly directional beam 'seeded' by upstream sound. The jet is issuing at a speed $0{\cdot}9c_0$.